LU Factorization (or encoding Gaussian Elimination)

Given a square matrix A, factor it as $A = LU$, where L = lower triangular and U = upper triangular

Ex: $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = LU$ (Check)

Note:
- LU factorizations are not unique
- LU factorizations don't always exist (more below)

We can use LU Factorizations to solve systems $Ax = b$.

Rewrite $Ax = b$ as $L\underbrace{Ux}_{=c} = b$ and then:

1. Solve $Lc = b$ for c using forward sub.
2. Solve $Ux = c$ for x using backward sub

} very quick for a computer

Ex: Solve
$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 4x_1 + 9x_2 - 3x_3 &= 8 \\ -2x_1 - 3x_2 + 7x_3 &= 10 \end{aligned} \iff \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

We know $\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

1. Solve $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} c = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$ $\left(\text{or } \begin{aligned} c_1 &= 2 \\ 2c_1 + c_2 &= 8 \\ -c_1 + c_2 + c_3 &= 10 \end{aligned}\right)$ $\xrightarrow{\text{forward substitution}}$ $\begin{aligned} c_1 &= 2 \\ c_2 &= 4 \\ c_3 &= 8 \end{aligned}$

2. Solve $\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$ $\xrightarrow{\text{backward sub}}$ $x_1 = -1,\ x_2 = 2,\ x_3 = 2$

Solution

Upshot of using this method

The computationally intensive portion of this process is factoring A.

Once you do it, you (ie. a computer) can very quickly solve $Ax=b$ for any b, since you do not need to compute $A=LU$ again

The Gauss-Jordan algorithm, on the other hand, requires you to re-run the entire process when b changes.

Finding an LU factorization

First, we interpret the elimination method as matrix multiplication.

The elimination matrix E_{ij} with multiplier l_{ij} is the matrix obtained from I by replacing the (i,j)-th entry by $-l_{ij}$

The product $E_{ij}A$ replaces Row$i$ of A with (Row$i$ $-$ l_{ij}Row$j$)

Ex: Let $E_{21} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ ($l_{21}=-2$), $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Then $E_{21}A = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}$ (replaced R2 with R2+2R1)

Note: E_{ij}^{-1} is the same as E_{ij} but w/ $-l_{ij}$ replaced by l_{ij}

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

E_{32} $\quad$ E_{32}^{-1}

(you'll think about this more in recitation)

To do an LU factorization, we perform eliminations until we get an upper triangular matrix U:

$$E_{i_n j_n} \cdots E_{i_2 j_2} E_{i_1 j_1} A = U$$

Key feature: Each of these elimination matrices is lower triangular

Now multiply both sides by $(E_{i_n j_n} \cdots E_{i_1 j_1})^{-1}$ to get:

$$A = (E_{i_n j_n} \cdots E_{i_1 j_1})^{-1} U = LU$$

Since the elimination matrices are lower triangular,

so is $L = (E_{i_n j_n} \cdots E_{i_1 j_1})^{-1} = E_{i_1 j_1}^{-1} \cdots E_{i_n j_n}^{-1}$

Ex: Find an LU factorization of $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$

First reduce A to U and record elimination matrices

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \xrightarrow[\text{Replace R3 w/ R3+R1}]{\substack{E_{21} \\ \text{Replace R2 w/ R2-2R1}}} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow[E_{32}]{\text{Replace R3 w/ R3-R2}} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = U$$

E_{31}

So $E_{32} E_{31} E_{21} A = U$

$\Rightarrow A = (E_{32} E_{31} E_{21})^{-1} U$

$\Rightarrow L = (E_{32} E_{31} E_{21})^{-1} = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

So $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ (Check by multiplying)

What if putting A in U.T. form requires row swaps?

$PA=LU$ Factorization

Swapping two rows of a matrix corresponds to multiplication by a permutation matrix:

P_{ij} is obtained from I by swapping the i^{th} and j^{th} rows.

Ex: For a 3×3, $P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Performing $P_{ij} \cdot A$ swaps (permutes) the i and j rows of A.

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 5 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$

In putting a matrix in U.T form we can first do the necessary row swaps and then do the eliminations.

Let P = product of all row swaps.

E = product of all eliminations

Then $EPA = U$ (upper triangular)

$\Rightarrow$ $PA = LU$ (where $L = E^{-1}$)

Ex: Find a $PA=LU$ factorization of $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{bmatrix}$ $\xrightarrow{\text{Swap } R2 \leftrightarrow R3}$ $\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 1 & 2 & 1 \end{bmatrix}$ $\xrightarrow[\text{Replace } R3 \text{ w/ } R3-R1]{\text{Replace } R2 \text{ w/ } R2-2R1}$ $\begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 9 \\ 0 & 0 & 3 \end{bmatrix} = U$

Here, $E = E_{31}E_{21}$ and $P = P_{23}$

So $L = E_{21}^{-1}E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Hence $PA=LU$.

Check: Verify $PA=LU$.

$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 1 & 2 & 1 \end{bmatrix}$

$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 9 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 1 & 2 & 1 \end{bmatrix}$

Unless otherwise specified (eg. on HW or exams), from now on, you can use Julia to do RREF for us